

*General Procedure — Separation of Variables

Solving steady-state heat conduction equation with
one nonhomogeneous boundary condition.

- ① Assume the separation of temperature function
 and reduce the original PDE to multiple ODEs.

$$\text{e.g.: } \begin{cases} X''(x) - \mu X(x) = 0 \\ Y''(y) + \mu Y(y) = 0 \end{cases}$$

- ② Solve ODEs with homogeneous boundary conditions.
 The solution involves the determination of eigenvalues
 and associated eigenfunctions.

$$\text{e.g.: } \lambda_m - X_m(x), Y_m(y)$$

- ③ Make the final solution the linear superposition of the
 elementary solutions (eigenfunctions) with unknown coefficients.

$$\text{e.g.: } T(x,y) = \sum_m C_m X_m(x) Y_m(y)$$

- ④ Determine unknown coefficients using the nonhomogeneous
 boundary condition. — Applying orthogonal property
 of eigenfunctions.

e.g.: orthogonal set $X_m(x)$,

$$\int_{\text{interval}} X_m(x) X_n(x) dx = \begin{cases} 0 & (m \neq n) \\ N_m & (m=n) \end{cases}$$

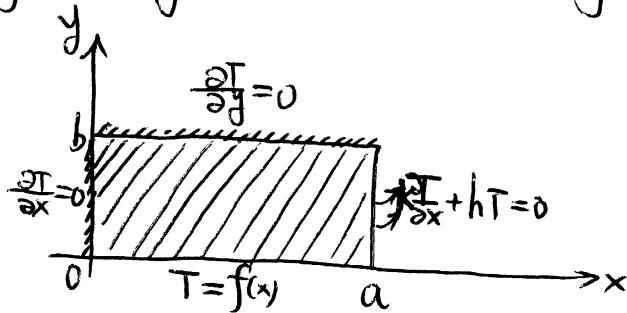
with

$$N_m = \int_{\text{interval}} X_m^2(x) dx$$

(normalization integral)

* Example 2.

Consider Steady-State temperature distribution $T(x, y)$ in a rectangular region, without heat generation, as follows:



The complete problem:

$$\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2} = 0$$

B.C.

$$\left\{ \begin{array}{l} \left. \frac{\partial T}{\partial x} \right|_{x=0} = 0 \\ \left. K \frac{\partial T}{\partial x} + hT \right|_{x=a} = 0 \\ \left. T \right|_{y=0} = f(x) \\ \left. \frac{\partial T}{\partial y} \right|_{y=b} = 0 \end{array} \right.$$

← Nonhomogeneous.

① Separation of $T(x, y)$

Assume: $T(x, y) = X(x)Y(y)$

then: $\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$

i.e.: $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu$ (not a function of x, y)

$$\text{or, } \begin{cases} \underline{X}''(x) - \mu \underline{X}(x) = 0 \\ \underline{Y}''(y) + \mu \underline{Y}(y) = 0 \end{cases}$$

The homogeneous boundary conditions can also be reduced:

$$\begin{cases} \frac{\partial T}{\partial x} \Big|_{x=0} = 0 \Rightarrow \frac{d\underline{X}}{dx} \Big|_{x=0} = 0 \\ k \frac{\partial T}{\partial x} \Big|_{x=a} + hT \Big|_{x=a} = 0 \Rightarrow \frac{d\underline{X}}{dx} \Big|_{x=a} + H\underline{X} \Big|_{x=a} = 0 \quad (\text{Note: } H \equiv \frac{h}{k}) \end{cases}$$

$$\left. \frac{\partial T}{\partial y} \right|_{y=b} = 0 \Rightarrow \frac{d\underline{Y}}{dy} \Big|_{y=b} = 0$$

② Solving reduced ODEs.

There are three cases: $\mu > 0$, $\mu = 0$, $\mu < 0$

(1) For $\mu > 0$ Let $\mu = \lambda^2$

$$\text{Therefore: } \begin{cases} \underline{X}''(x) - \lambda^2 \underline{X}(x) = 0 \\ \underline{X}(x) = A \cosh \lambda x + B \sinh \lambda x \end{cases}$$

$$\begin{cases} \underline{Y}''(y) + \lambda^2 \underline{Y}(y) = 0 \\ \underline{Y}(y) = C \cos \lambda y + D \sin \lambda y \end{cases}$$

And:

$$T(x, y) = (A \cosh \lambda x + B \sinh \lambda x)(C \cos \lambda y + D \sin \lambda y)$$

Impose B.C. $\frac{\partial T}{\partial x} \Big|_{x=0} \rightarrow \frac{d\underline{X}}{dx} \Big|_{x=0} = 0$

$$\begin{aligned} \frac{d\underline{X}}{dx} \Big|_{x=0} &= [A \lambda \sinh \lambda x + B \lambda \cosh \lambda x] \Big|_{x=0} \\ &= B \lambda = 0 \end{aligned}$$

Therefore, $\boxed{B=0}$ and $\boxed{\underline{X}(x) = A \cosh \lambda x}$

Imposing B.C. $\left. k \frac{\partial T}{\partial x} \right|_{x=a} + hT \Big|_{x=a} = 0 \rightarrow \left. \frac{dX}{dx} \right|_{x=a} + H \bar{X} \Big|_{x=a} = 0$

$$\begin{aligned} \left. \frac{dX}{dx} \right|_{x=a} + H \bar{X} \Big|_{x=a} &= A \lambda \sinh \lambda x \Big|_{x=a} + H A \cosh \lambda x \Big|_{x=a} \\ &= A (\lambda \sinh \lambda a + H \cosh \lambda a) = 0 \end{aligned}$$

therefore, $\boxed{A=0}$ and $\boxed{\bar{X}(x)=0}$ not meaningful solution

Conclusion: μ cannot be greater than 0!

(2) For $\mu=0$

Therefore:

$$\begin{cases} \bar{X}''(x) = 0 \\ \bar{X}(x) = Ax + B \end{cases}$$

$$\begin{cases} \bar{Y}''(y) = 0 \\ \bar{Y}(y) = Cy + D \end{cases}$$

and: $T(x, y) = (Ax + B)(Cy + D)$

Imposing B.C. $\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0 \rightarrow \left. \frac{dX}{dx} \right|_{x=0} = 0$

$$\left. \frac{dX}{dx} \right|_{x=0} = A = 0$$

i.e. $\boxed{A=0}$ and $\boxed{\bar{X}(x)=B}$

Imposing B.C. $\left. k \frac{\partial T}{\partial x} \right|_{x=a} + hT \Big|_{x=a} = 0 \rightarrow \left. \frac{dX}{dx} \right|_{x=a} + H \bar{X} \Big|_{x=a} = 0$

$$\left. \frac{dX}{dx} \right|_{x=a} + H \bar{X} \Big|_{x=a} = 0 + HB = 0$$

i.e. $\boxed{B=0}$ and $\boxed{\bar{X}(x)=0}$

Conclusion: μ cannot be 0!

3) For $\mu < 0$ Let $\mu = -\lambda^2$ ($\lambda > 0$)

Therefore:

$$\begin{cases} \bar{X}'(x) + \lambda^2 \bar{X}(x) = 0 \\ \bar{X}(x) = A \cos \lambda x + B \sin \lambda x \end{cases}$$

$$\begin{cases} \bar{Y}''(y) - \lambda^2 \bar{Y}(y) = 0 \\ \bar{Y}(y) = C \cosh \lambda y + D \sinh \lambda y \end{cases}$$

and:

$$T(x, y) = (\bar{A} \cos \lambda x + \bar{B} \sin \lambda x)(\bar{C} \cosh \lambda y + \bar{D} \sinh \lambda y)$$

Impose B.C. $\frac{\partial T}{\partial x} \Big|_{x=0} \rightarrow \frac{d \bar{X}}{dx} \Big|_{x=0} = 0$

$$\begin{aligned} \frac{d \bar{X}}{dx} \Big|_{x=0} &= -A\lambda \sin \lambda x \Big|_{x=0} + B\lambda \cos \lambda x \Big|_{x=0} \\ &= 0 + B\lambda = 0 \end{aligned}$$

i.e., $\boxed{B=0}$ and $\boxed{\bar{X}(x) = A \cos \lambda x}$

Impose B.C. $K \frac{\partial T}{\partial x} + H T \Big|_{x=a} = 0 \rightarrow \frac{d \bar{X}}{dx} \Big|_{x=a} + H \bar{X} \Big|_{x=a} = 0$

$$\begin{aligned} \frac{d \bar{X}}{dx} \Big|_{x=a} + H \bar{X} \Big|_{x=a} &= -A\lambda \sin \lambda x \Big|_{x=a} + H A \cos \lambda x \Big|_{x=a} \\ &= -A\lambda \sin \lambda a + H A \cos \lambda a = 0 \end{aligned}$$

i.e. $A(-\lambda \sin \lambda a + H \cos \lambda a) = 0$

There are two possibilities: $A=0$ or $-\lambda \sin \lambda a + H \cos \lambda a = 0$.

If $A=0 \Rightarrow \bar{X}(x)=0$ not a meaningful solution.

So, we must have: $-\lambda \sin \lambda a + H \cos \lambda a = 0$

i.e. $\boxed{\cot(\lambda a) = \frac{\lambda}{H}}$

Therefore, λ can only take certain values that are the positive roots of,

$$\cot \lambda_m a = \frac{\lambda_m}{H}$$

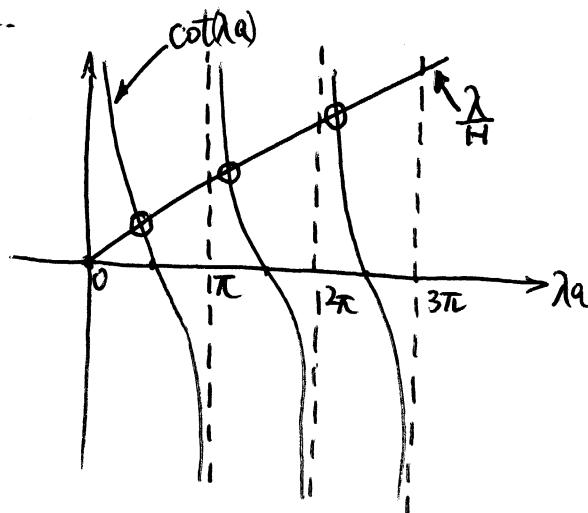
(equivalent: $\lambda_m \tan \lambda_m a = H$)

and,

$$X_m(x) = A_m \cos \lambda_m x$$

$m=1, 2, 3, \dots$

On the right is the geometrical representation of the roots of $\cot \lambda a = \frac{\lambda}{H}$ (intersections)



Imposing B.C. $\left. \frac{\partial Y}{\partial y} \right|_{y=b} = 0 \rightarrow \left. \frac{dY}{dy} \right|_{y=b} = 0$

$$\begin{aligned} \frac{dY}{dy} &= C \lambda \sinh \lambda y \Big|_{y=b} + D \lambda \cosh \lambda y \Big|_{y=b} \\ &= C \lambda \sinh \lambda b + D \lambda \cosh \lambda b = 0 \end{aligned}$$

i.e. $D = -C \frac{\sinh \lambda b}{\cosh \lambda b}$

and $Y(y) = C \left(\cosh \lambda y - \frac{\sinh \lambda b}{\cosh \lambda b} \sinh \lambda y \right)$
or, $Y(y) = C' \cosh \lambda (b-y)$

③ Make the final solution.

$$T_m(x, y) = A_m \cosh \lambda_m(b-y) \cdot \cos \lambda_m x, \text{ with } \lambda_m \text{ defined by } \cot \lambda_m a = \frac{\lambda_m}{H}.$$

$$T(x, y) = \sum_{m=1}^{\infty} A_m \cosh \lambda_m(b-y) \cos \lambda_m x$$

(4) Determine unknown coefficients.

Applying Nonhomogeneous boundary conditions: $T|_{y=0} = f(x)$.

$$T|_{y=0} = \sum_{m=1}^{\infty} A_m \cosh \lambda_m b \cos \lambda_m x = f(x)$$

$$\int_0^a \left[\sum_{m=1}^{\infty} A_m \cosh \lambda_m b \cos \lambda_m x \right] \cos \lambda_n x dx = \int_0^a f(x) \cos \lambda_n x dx$$

Using orthogonal property of the eigenfunction:

$$\int_0^a \cos \lambda_m x \cdot \cos \lambda_n x dx = \begin{cases} 0 & \text{for } m \neq n \\ N_m & \text{for } m = n \end{cases}$$

Therefore, (for $n=m$)

$$\int_0^a A_m \cosh \lambda_m b \cos \lambda_m x \cdot \cos \lambda_m x dx = \int_0^a f(x) \cos \lambda_m x dx$$

$$A_m \cosh \lambda_m b \underbrace{\int_0^a \cos^2 \lambda_m x dx}_{N_m} = \int_0^a f(x) \cos \lambda_m x dx$$

It can be shown that, (next page)

$$N_m = \int_0^a \cos^2 \lambda_m x dx = \frac{1}{2} \left(a + \frac{H}{\lambda_m^2 + H^2} \right)$$

$$\text{So: } A_m = \frac{2(\lambda_m^2 + H^2)}{a(\lambda_m^2 + H^2) + H} \cdot \frac{1}{\cosh \lambda_m b} \int_0^a f(x) \cos \lambda_m x dx$$

$$T(x, y) = \sum_{m=1}^{\infty} \frac{2(\lambda_m^2 + H^2)}{a(\lambda_m^2 + H^2) + H} \cdot \frac{\cosh \lambda_m (b y)}{\cosh \lambda_m b} \cos \lambda_m x \int_0^a f(x) \cos \lambda_m x dx$$

Finding: $N_m = \int_0^a \cos^2 \lambda_m x dx$

$$\begin{aligned}
 N_m &= \int_0^a \cos^2 \lambda_m x dx \\
 &= \int_0^a \frac{1 + \cos 2\lambda_m x}{2} dx \quad (\cos^2 \theta = \frac{1 + \cos 2\theta}{2}) \\
 &= \int_0^a \frac{1}{2} dx + \int_0^a \frac{\cos 2\lambda_m x}{2} dx \\
 &= \frac{a}{2} + \frac{1}{2} \cdot \frac{\sin 2\lambda_m a}{2\lambda_m} \\
 &= \frac{1}{2} \left[a + \frac{\sin \lambda_m a \cdot \cos \lambda_m a}{\lambda_m} \right] \quad (\sin 2\theta = 2 \sin \theta \cos \theta)
 \end{aligned}$$

Because λ_m satisfies $\lambda_m \tan \lambda_m a = H$,

$$\text{i.e., } \frac{\sin \lambda_m a}{\cos \lambda_m a} = \frac{H}{\lambda_m}$$

$$\begin{cases} \text{So } (1) \sin \lambda_m a \cdot \cos \lambda_m a = \frac{H}{\lambda_m} \cos^2 \lambda_m a \\ (2) \frac{\sin^2 \lambda_m a}{\cos^2 \lambda_m a} = \frac{1 - \cos^2 \lambda_m a}{\cos^2 \lambda_m a} = \frac{H^2}{\lambda_m^2} \Rightarrow \cos^2 \lambda_m a = \frac{1}{1 + \frac{H^2}{\lambda_m^2}} \end{cases}$$

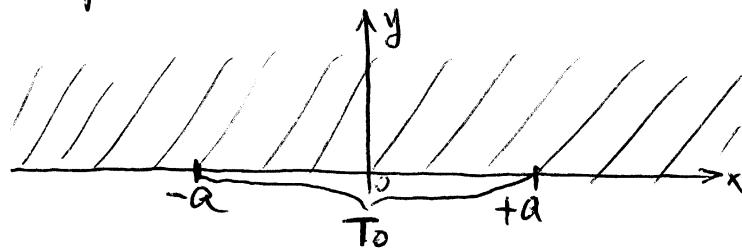
$$\begin{aligned}
 \text{Therefore, } N_m &= \frac{1}{2} \left[a + \frac{H \cos^2 \lambda_m a}{\lambda_m^2} \right] \\
 &= \frac{1}{2} \left[a + \frac{H}{\lambda_m^2} \left(\frac{1}{1 + \frac{H^2}{\lambda_m^2}} \right) \right]
 \end{aligned}$$

i.e.

$$\underbrace{N_m = \frac{1}{2} \left(a + \frac{H}{\lambda_m^2 + H^2} \right)}_{\text{Final Answer}}$$

* Example 3.

Consider steady-state temperature distribution $T(x, y)$ in a half-plane ($y: 0 \rightarrow \infty$, $x: -\infty \rightarrow +\infty$). Portion of the x -axis between $\pm a$ is maintained at T_0 , the rest of the x -axis has temperature $T=0$.



Complete Problem:

$$\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2} = 0$$

B.C. $\left. T \right|_{y=0} = \begin{cases} T_0 & |x| < a \\ 0 & |x| > a \end{cases}$

$$\left. T \right|_{y \rightarrow +\infty}$$

$$\left. T \right|_{x \rightarrow -\infty}$$

$$\left. T \right|_{x \rightarrow +\infty}$$

① Assume: $T(x, y) = X(x) Y(y)$

Then: $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu \text{ (a const.)}$

i.e. $\begin{cases} X''(x) - \mu X(x) = 0 \\ Y''(y) + \mu Y(y) = 0 \end{cases}$

There are three possible cases: $\mu > 0$, $\mu = 0$, $\mu < 0$.

② Solving ODEs

(1) $\mu > 0$ Let $\mu = \lambda^2$

$$\begin{cases} X''(x) - \lambda^2 X(x) = 0 \\ X(x) = Ae^{\lambda x} + Be^{-\lambda x} \end{cases}$$

$$\begin{cases} Y''(y) + \lambda^2 Y(y) = 0 \\ Y(y) = C \cos \lambda y + D \sin \lambda y \end{cases}$$

therefore: $T(x, y) = (Ae^{\lambda x} + Be^{-\lambda x})(C \cos \lambda y + D \sin \lambda y)$

Impose B.C. $T|_{x \rightarrow -\infty} = \text{finite} \Rightarrow B = 0$

Impose B.C. $T|_{x \rightarrow +\infty} = \text{finite} \Rightarrow A = 0$

So: $T(x, y) = 0$ not a meaningful solution.

Conclusion: μ cannot be greater than 0 !

(2) $\mu = 0$

$$\begin{cases} X''(x) = 0 \\ X(x) = Ax + B \end{cases}$$

$$\begin{cases} Y''(y) = 0 \\ Y(y) = Cy + D \end{cases}$$

therefore: $T(x, y) = (Ax + B)(Cy + D)$

Impose B.C. $T|_{x \rightarrow \infty} = \text{finite} \Rightarrow A = 0$

Impose B.C. $T|_{y \rightarrow \infty} = \text{finite} \Rightarrow C = 0$

So: $T(x, y) = BD = \text{const.}$ not a meaningful solution

Conclusion: μ cannot be 0 !

(3) $\mu < 0$ Let $\mu = -\lambda^2$ ($\lambda > 0$)

$$\begin{cases} X''(x) + \lambda^2 X(x) = 0 \\ X(x) = A \cos \lambda x + B \sin \lambda x \end{cases}$$

$$\begin{cases} Y''(y) - \lambda^2 Y(y) = 0 \\ Y(y) = C e^{\lambda y} + D e^{-\lambda y} \end{cases}$$

therefore: $T(x, y) = (A \cos \lambda x + B \sin \lambda x)(C e^{\lambda y} + D e^{-\lambda y})$

Imposing B.C. $T|_{y \rightarrow 0} = \text{finite} \Rightarrow C = 0$

so: $T(x, y) = (A \cos \lambda x + B \sin \lambda x)e^{-\lambda y}$

Using symmetric property of the problem.

$$T(x, y) = T(-x, y)$$

(according to B.C. at $y=0$)

so: $T(x, y) = A \cos \lambda x e^{-\lambda y}$ ($B=0$)

There is no further condition to restrict λ , so we have a continuous family of solutions (instead of discrete values):

$$T_\lambda(x, y) = A(\lambda) e^{-\lambda y} \cos \lambda x$$

(and the "addition" needs to be changed to "integration" over all values of λ for linear superposition.)

③ Make the final solution:

$$T(x, y) = \int_0^\infty T_\lambda(x, y) d\lambda = \int_0^\infty A(\lambda) e^{-\lambda y} \cos \lambda x d\lambda$$

(4) Determine the unknown coefficients.

Applying the nonhomogeneous BC. $T|_{y=0} = \begin{cases} T_0 & |x| < a \\ 0 & |x| > a \end{cases}$

$$T|_{y=0} = \int_0^{\infty} A(\lambda) \cos \lambda x d\lambda = f(x) = \begin{cases} T_0 & |x| < a \\ 0 & |x| > a \end{cases}$$

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi} \int_0^{\infty} f(x) \cos \lambda x dx \\ &= \frac{2}{\pi} \int_0^a T_0 \cos \lambda x dx \\ &= \frac{2}{\pi} T_0 \frac{\sin \lambda a}{\lambda} \end{aligned}$$

Therefore:

$$T(x, y) = \frac{2}{\pi} T_0 \int_0^{\infty} \frac{\sin \lambda a \cos \lambda x}{\lambda} e^{-\lambda y} d\lambda$$

Note: Fourier integral pairs:

✓ even function $f(x)$:

$$\begin{cases} f(x) = \int_0^{\infty} A(\lambda) \cos \lambda x d\lambda \\ A(\lambda) = \frac{2}{\pi} \int_0^{\infty} f(x) \cos \lambda x dx \end{cases}$$

✓ odd function $f(x)$:

$$\begin{cases} f(x) = \int_0^{\infty} B(\lambda) \sin \lambda x d\lambda \\ B(\lambda) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin \lambda x dx \end{cases}$$